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SIMULTANEOUS STATIC AND KINEMATIC INDETERMINACY OF SPACE TRUSSES WITH CYCLIC SYMMETRY

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Abstract—This paper examines the structural rigidity of pin-jointed space trusses with cyclic symmetry. Although the necessary condition of static and kinematic determinacy, the so-called *Maxwell's rule*, is satisfied, the structures are both statically and kinematically indeterminate. The physical meaning of the static and kinematic indeterminacy is presented. In the case of both statically and kinematically indeterminate cylindrical trusses having planes of mirror symmetry, it is proved that although the cylindrical truss consists of individually both unstable and redundant "rings", the degree of static and kinematic indeterminacy is independent of the number of the "rings".

1. INTRODUCTION

In structural engineering the so-called *Maxwell's rule* [3] is well-known: a space truss having b straight bars and j frictionless pin joints is, in general, simply stiff if $b = 3j - 6$ [7, 4, 8]. If the six bars connecting the truss to a foundation are included in b then $b = 3j$. This rule, however, is only a necessary condition for simple stiffness, or more correctly, kinematic determinacy of the framework. This *algebraic* condition is not sufficient to determine whether a truss is stiff or not. Moreover the knowledge of the *topological* properties of a truss, i.e. the graph of a truss, is still not sufficient for this. The stiffness can be ascertained only by the *geometric*, i.e. the metric properties of a truss, by the help of the "geometric" matrix [10, 11] ("compatibility matrix" [6]) of the undeformed truss, or by the help of its transpose, the "equilibrium matrix". Maxwell's rule is a necessary condition for static determinacy of a space truss. The static determinacy also is a function only of the geometry of the truss. In general, when a space truss has very many bars (and disregarding some trivial cases) it is not easy to ascertain the static and kinematic determinacy or indeed to discover the physical meaning of static and kinematic indeterminacy.

In this study a simple type of space truss will be considered for which the joints lie on a surface of revolution and on planes which are orthogonal to the axis of revolution, so that the bars lying on the same plane form a regular n-gon (Figs. 1 and 8). Each joint is connected by two bars to the n-gon which is above it and by two bars to the n-gon which is below it. The whole network is such that the central axis is an axis of n-fold rotational symmetry. The truss is considered to be composed of straight bars connected to one another and to a foundation by ideal frictionless spherical joints; and it may be subjected to external forces applied at the joints.

Let us consider first a reticulated cylinder shown in Fig. 1(a). In this case Maxwell's rule is valid and the truss is indeed both statically and kinematically determinate. For the determination of the forces in the bars of this truss, effective methods are known (see, e.g. [2]). But the state of determinacy of the truss may change when, keeping the topological properties of the original truss, the cylinder is transformed into a framework (Fig. 1b) which also has a plane of mirror symmetry passing through its axis of n-fold rotational symmetry. Maxwell's rule is valid in this case also. This network is obtained from that of Fig. 1(a) by rotating every n-sided polygon in the same direction by angle π/n measured from the adjacent polygon under it. In this paper the circumstances in which the reticulated cylinder will not be statically and kinematically determined will be analyzed. The physical meaning of the static and kinematic indeterminacy will be discussed.

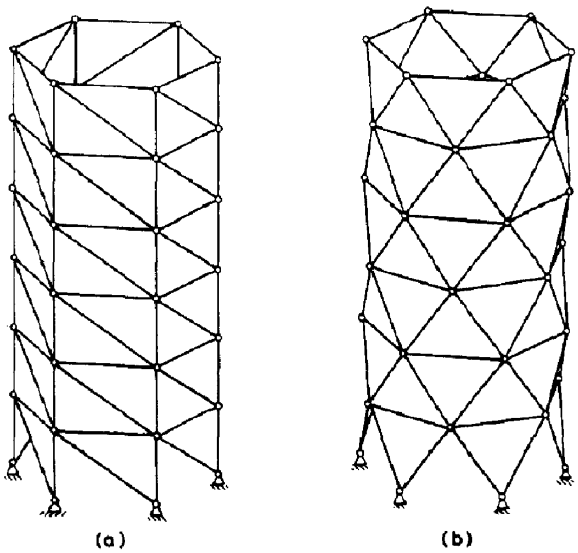


Fig. 1. Reticulated cylinders with (a) horizontal, vertical and inclined bars, and (b) horizontal and inclined bars in symmetrical arrangement.

2. STATIC AND KINEMATIC PROPERTIES OF A RING

In this paper, the union of an n-sided polygon and the bars connecting it to the adjacent polygon will be called a ring. Consider a typical ring of the reticulated cylinder shown in Fig. 1(a). Rotate the upper n-gon anti-clockwise by an angle θ measured relative to the lower one. During this rotation the length of the vertical and inclined bars will, in general, change. The problem which then arises is, whether there exists a value or values of the angle θ for which the matrix determining the equilibrium of the ring is singular.

Let us number the upper joints of the ring clockwise from 1 to n and the lower ones in the same manner from $n+1$ to $2n$ as shown in Fig. 2. Let the radius of the cylinder be unity and let the height of the ring be m. Let us now write the equations of equilibrium of the forces in each

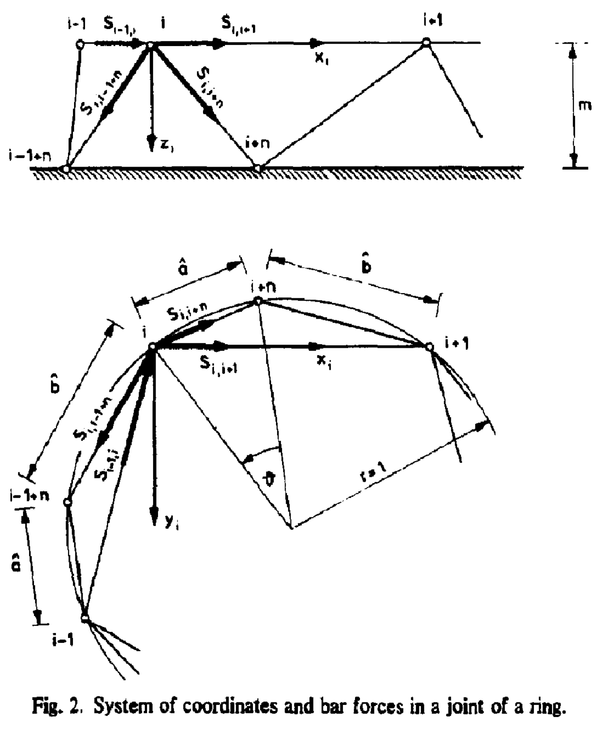


Fig. 2. System of coordinates and bar forces in a joint of a ring.

upper joint. It will be convenient for this purpose to use a right-hand orthogonal system of coordinates which is such that the origin is in the joint, the x axis lies in a side of the n-gon and its positive direction with respect to the axis of revolution is clockwise, the y axis lies in the plane of the n-gon and its positive direction is towards the inside of the polygon, and the z axis has its positive direction downwards. If the cyclic order of succession of the forces in the bars is fixed as $S_{i,i+1}$; $S_{i,i-1+n}$; $S_{i,i+n}$ and the equations of the resolved forces are written in the order x_i, y_i, z_i then the coefficient matrix of the equations of equilibrium, i.e. the "equilibrium matrix" of the ring, will be the following hyper-cyclic matrix:

$$\mathbf{L} = \begin{array}{c} \begin{array}{cccc} \underset{\sim}{1} & \underset{\sim}{2} & \cdots & \underset{\sim}{n} \end{array} \\ \begin{bmatrix} \mathbf{A} & & & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & & \\ & \mathbf{B} & \mathbf{A} & \\ & & \ddots & \ddots \\ & & \mathbf{B} & \mathbf{A} \end{bmatrix} \end{array}, \tag{1}$$

in which

$$\mathbf{A} = \begin{bmatrix} -1 & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} g & 0 & 0 \\ h & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

where

$$a = \frac{\hat{b}}{l_b} \sin\left[\frac{(n-4)\pi}{2n} + \frac{\theta}{2}\right], \tag{3}$$

$$b = -\frac{\hat{a}}{l_a} \cos\left(\frac{\pi}{n} - \frac{\theta}{2}\right), \tag{4}$$

$$c = -\frac{\hat{b}}{l_b} \cos\left[\frac{(n-4)\pi}{2n} + \frac{\theta}{2}\right], \tag{5}$$

$$d = \frac{\hat{a}}{l_a} \sin\left(\frac{\pi}{n} - \frac{\theta}{2}\right), \tag{6}$$

$$e = -\frac{m}{l_b}, \tag{7}$$

$$f = -\frac{m}{l_a}, \tag{8}$$

$$g = -\cos\frac{2\pi}{n}, \tag{9}$$

$$h = \sin\frac{2\pi}{n}. \tag{10}$$

Here $\hat{a}$ and $\hat{b}$ denote the projected length of the bars (and also of the corresponding bars according to the rotational symmetry) connecting the ith, $i+n$th and $i+n$th, $i+1$st joints, respectively, in the $x-y$ plane, while l_a and l_b denote their true length. The matrix $\mathbf{L}$ is singular, if its determinant, denoted by det $\mathbf{L}$, is equal to zero. Expanding the determinant and setting it equal to zero we obtain

$$\det \mathbf{L} = f^n \left\{ \left(c - d\frac{e}{f}\right)^n - (-1)^n \left[h\left(a - b\frac{e}{f}\right) - g\left(c - d\frac{e}{f}\right)\right]^n \right\} = 0.$$

Since $f \neq 0$ it follows that

$$\left(c - d\frac{e}{f}\right)^n - (-1)^n \left[h\left(a - b\frac{e}{f}\right) - g\left(c - d\frac{e}{f}\right)\right]^n = 0. \tag{11}$$

By substituting the expressions (3)–(10) into eqn (11) we find

$$\left[\sin\left(\frac{2\pi}{n}-\theta\right)\right]^n-[-\sin\theta]^n=0. \tag{12}$$

The character of the solutions of this equation depends on the parity of n, the number of the sides of the regular polygon. If n is odd, then eqn (12) has two real solutions θ:

$$\theta_1=\frac{(n+2)\pi}{2n}\pm 2k\pi \qquad k=0,1,2,\ldots, \tag{13}$$

$$\theta_2=\frac{(3n+2)\pi}{2n}\pm 2k\pi \qquad k=0,1,2,\ldots. \tag{14}$$

Solutions (13) and (14) determine a right-hand screwed position and a left-hand screwed position of the ring, respectively. In the case $n=3$ these two trusses are shown in Fig. 3. It should be noted that the properties of the truss sketched in Fig. 3, conceived as a polyhedron, were analysed by Wunderlich[12].

If n is even, then (13) and (14) will be again solutions of eqn (12) but there exist also two other real solutions of eqn (12):

$$\theta_3=\frac{\pi}{n}\pm 2k\pi \qquad k=0,1,2,\ldots, \tag{15}$$

$$\theta_4=\frac{(n+1)\pi}{n}\pm 2k\pi \qquad k=0,1,2,\ldots. \tag{16}$$

The solutions (13) and (14) determine again right- and left-handed screwed trusses, respectively, but the solutions (15) and (16) result in frameworks having planes of mirror symmetry. For the case $n=4$ these four trusses are shown in Fig. 4. In the case of trusses corresponding to the solutions (13), (14) and/or (13)–(16) the equilibrium matrix **L** of the ring is singular, and it can be proved that in each case the nullity of the matrix **L** is equal to one. (Nullity or defect of a

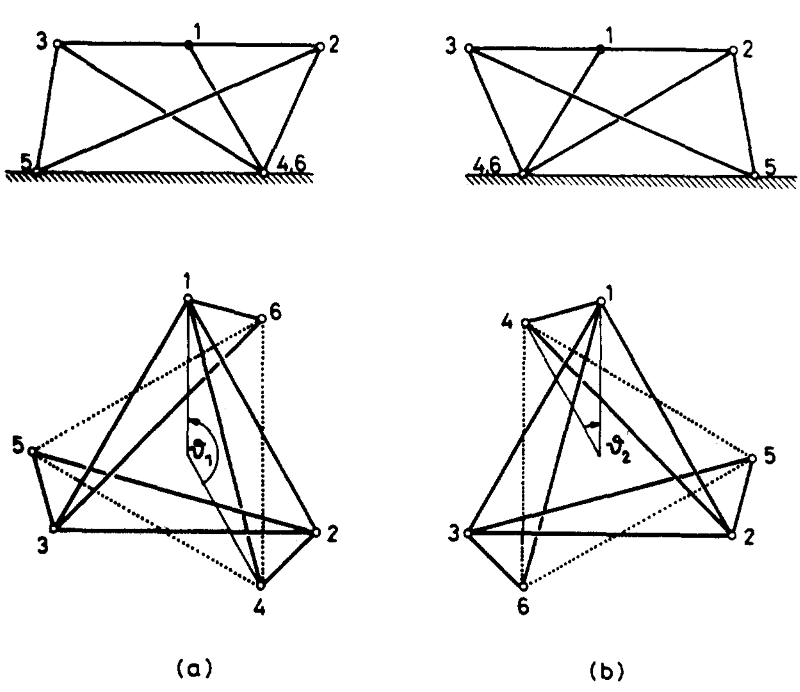


Fig. 3. Both statically and kinematically indeterminate triangular rings (a) in a right-hand screwed position and (b) in a left-hand screwed position.

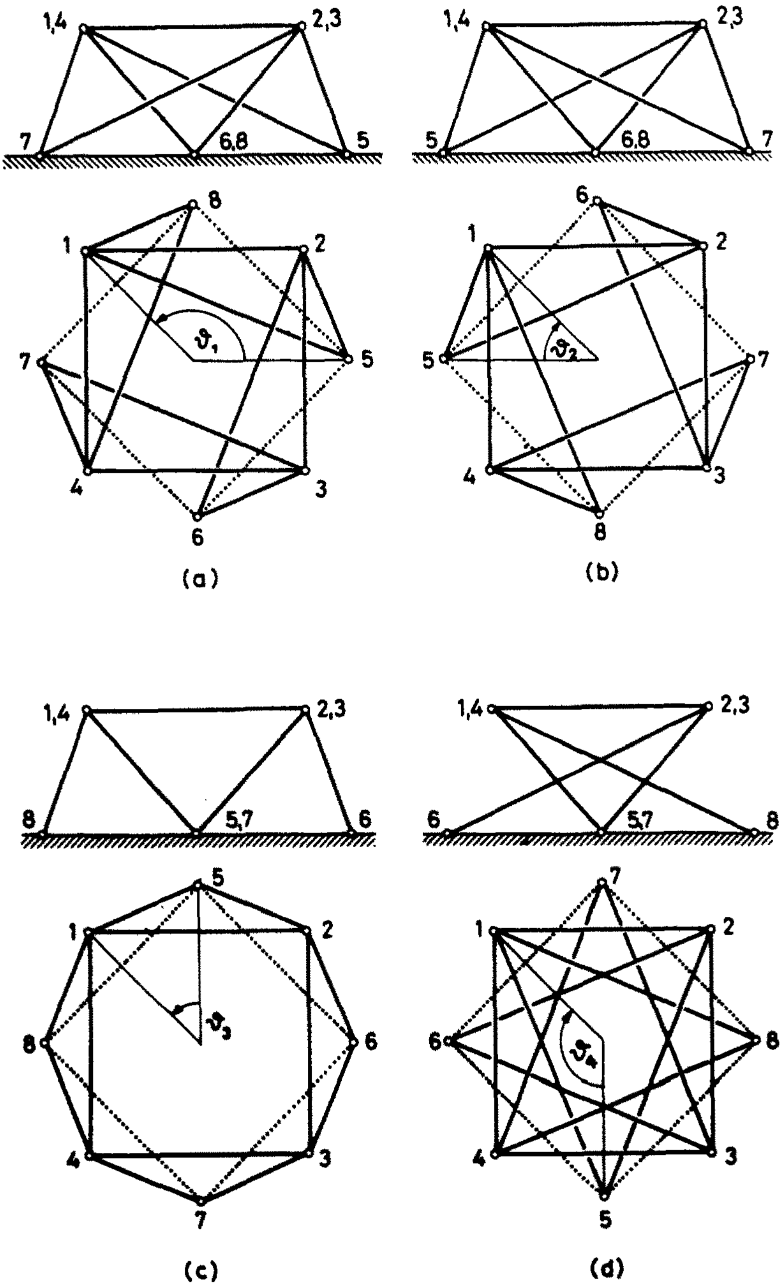


Fig. 4. Both statically and kinematically indeterminate quadrangular rings (a) in a right-hand screwed position, (b) in a left-hand screwed position, (c) and (d) in symmetrical arrangements.

square matrix is defined as a difference between the order and the rank of the matrix[1].) Since the matrix **L** is square, the singularity with nullity 1 means that the ring is both statically and kinematically indeterminate[10, 11] and the degree of the indeterminacy is 1. The single degree of static indeterminacy means that the ring can be in a *state of self-stress* in which the internal forces form a system of one parameter and the value of the parameter is arbitrary. The single degree of kinematic indeterminacy means that the ring is a mechanism of single degree of freedom: in the case of expressions (13), (14) it is an "*infinitesimal mechanism*" and in the case of expressions (15), (16) it is a "*large-displacement mechanism*". When n is even and the framework has planes of mirror symmetry, the joints of the ring can move in the radial direction. The even joints move towards the inside of the n-gon while the odd ones move towards the outside and vice versa. During this motion, the joints also leave the plane of the n-gon.

It should be mentioned that the solutions (13), (14) determine so-called "*Tensegrity*" structures which can be in a state of self-stress and are "free in space", i.e. not attached to a foundation[3, 9]. In this case the dotted lines in Figs. 3(a), (b) and 4(a), (b) are bars belonging to the structures.

3. SYMMETRICAL CYLINDRICAL TRUSS

In the remainder of the paper only the framework having planes of mirror symmetry, corresponding to (15), will be dealt with, and the properties of the reticulated cylinder which is constructed from m congruent rings by building on one another (Fig. 1b) will be analyzed. In this section, by the following analytical derivations, it will be proved that the degree of static and kinematic indeterminacy of the cylinder is always 1, independently of the number of the rings in the cylinder.

The analysis will be done by using a system of local coordinates. The notation will be different from that used in Section 2, but the scheme for numbering the joints will remain unchanged. To each joint, a system of non-orthogonal coordinates will be fitted, the three axes of which are in the direction of the bars. One of these axes lies in the "ring" direction and the two others go downward symmetrically (Fig. 5a). An external force-vector $\mathbf{P}_i$ loading the ith joint will be decomposed into components lying in direction of the coordinate unit vectors $\mathbf{e}_{i,i+1}$; $\mathbf{e}_{i,i+n}$; $\mathbf{e}_{i,i+n-1}$ as follows:

$$\mathbf{P}_i = P_i\mathbf{e}_i = P_{i,i+1}\mathbf{e}_{i,i+1} + P_{i,i+n}\mathbf{e}_{i,i+n} + P_{i,i+n-1}\mathbf{e}_{i,i+n-1}. \tag{17}$$

Then the equation of equilibrium of ith joint (Fig. 6) may be written:

$$\begin{aligned} &- S_{i-n,i}\mathbf{e}_{i-n,i} - S_{i-n+1,i}\mathbf{e}_{i-n+1,i} - S_{i-1,i}\mathbf{e}_{i-1,i} \\ &+ S_{i,i+1}\mathbf{e}_{i,i+1} + S_{i,i+n}\mathbf{e}_{i,i+n} + S_{i,i-1+n}\mathbf{e}_{i,i-1+n} = P_i\mathbf{e}_i \end{aligned} \tag{18}$$

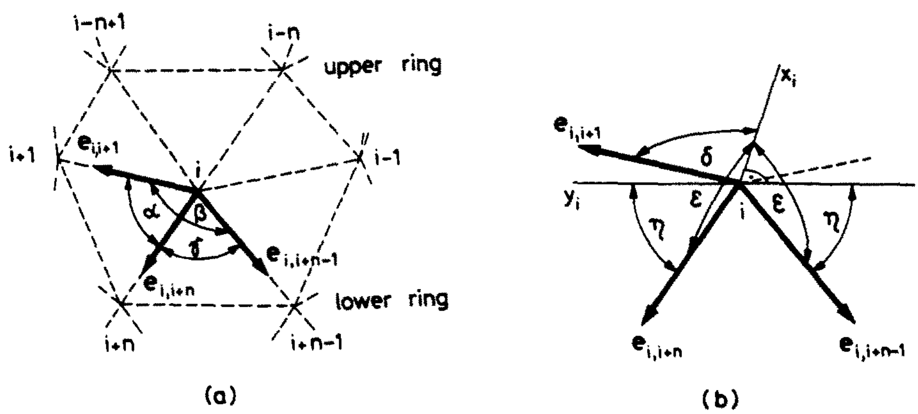


Fig. 5. System of coordinates in the ith joint; (a) arrangement of the coordinate axes, (b) angles between the different axes.

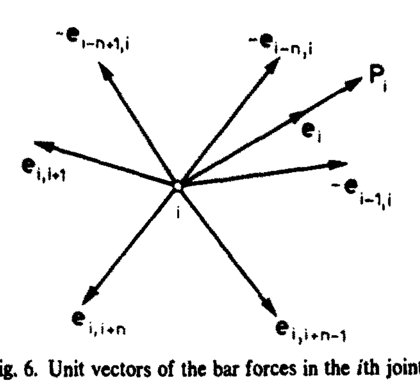


Fig. 6. Unit vectors of the bar forces in the ith joint.

where the symbols S with subscripts denote the magnitudes of the forces in the bars. Let α, β, γ denote the angles between the coordinate axes (Fig. 5a). Let x_i denote the straight line lying in the plane of the regular n-gon containing the ith joint, going through the ith joint and the centre of the n-gon, let y_i denote the straight line lying on the plane of the n-gon, intersecting the x_i straight line orthogonally in the ith joint. Let δ, ϵ, η denote the angles between the coordinate axes and straight lines x_i, y_i (Fig. 5b). If the cyclic order of succession of forces in bars and components of external forces is fixed as $i, i+1$; $i, i+n$; $i, i+n-1$ then eqns (18) ($i = 1, 2, \ldots, mn$) give the following set of equations of equilibrium (at the subscripts, it should be considered that numbering of joints shows a cyclic property):

$$\begin{array}{c} \begin{array}{cccccc} \underbrace{1} & \underbrace{2} & \underbrace{3} & \cdots & & \underbrace{m} \end{array} \\ \begin{bmatrix} \mathbf{L} & & & & & \\ \mathbf{N} & \mathbf{L} & & & & \\ & \mathbf{N} & \mathbf{L} & & & \\ & & \ddots & \ddots & & \\ & & & \mathbf{N} & \mathbf{L} & \\ & & & & \mathbf{N} & \mathbf{L} \end{bmatrix} \end{array} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \vdots \\ \mathbf{s}_{m-1} \\ \mathbf{s}_m \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \vdots \\ \mathbf{p}_{m-1} \\ \mathbf{p}_m \end{bmatrix}. \tag{19}$$

Here $\mathbf{L}$ is the equilibrium matrix of one ring, which may be written

$$\mathbf{L} = \begin{array}{c} \begin{array}{cccccc} \underbrace{1} & \underbrace{2} & \underbrace{3} & \cdots & & \underbrace{n} \end{array} \\ \begin{bmatrix} \mathbf{E} & & & & & \mathbf{C} \\ \mathbf{C} & \mathbf{E} & & & & \\ & \mathbf{C} & \mathbf{E} & & & \\ & & \ddots & \ddots & & \\ & & & \mathbf{C} & \mathbf{E} & \\ & & & & \mathbf{C} & \mathbf{E} \end{bmatrix} \end{array}, \tag{20}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ a & 0 & 0 \end{bmatrix}, \tag{21}$$

and $\mathbf{N}$ is the transfer matrix (the matrix of spread of forces), which may be written

$$\mathbf{N} = \begin{array}{c} \begin{array}{cccccc} \underbrace{1} & \underbrace{2} & \underbrace{3} & \cdots & & \underbrace{n} \end{array} \\ \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \\ & \mathbf{A} & \mathbf{B} & & & \\ & & \mathbf{A} & \mathbf{B} & & \\ & & & \ddots & \ddots & \\ & & & & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & & & & & \mathbf{A} \end{bmatrix} \end{array} \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & b & 0 \\ 0 & -c & 0 \\ 0 & d & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & -d \\ 0 & 0 & e \end{bmatrix}. \tag{23}$$

The symbols in matrices (21), (23) denote the following expressions:

$$a = \frac{\cos\alpha - \cos\beta}{1 - \cos\gamma}, \qquad b = 2\frac{\cos\epsilon}{\cos\delta},$$

$$c = 1 + \frac{\tan\delta\cos\epsilon}{\cos\eta}, \qquad d = \frac{\tan\delta\cos\epsilon}{\cos\eta}, \qquad e = -1 + \frac{\tan\delta\cos\epsilon}{\cos\eta}.$$

The symbols $\mathbf{s}_j$ and $\mathbf{p}_j$ in eqn (19) denote the hypervector composed of forces in bars of jth ring and the hypervector composed of components of external forces acting at the top of the jth ring, respectively, $(j = 1,2\ldots, m)$. *Let us denote the coefficient matrix of eqn* (19) *by* **M**. It follows from structure of matrix **M** that[1]

$$\det \mathbf{M} = (\det \mathbf{L})^m, \tag{24}$$

where det **M** denotes the determinant of matrix **M**. If n is odd, then $\det \mathbf{L} \neq 0$ and in consequence $\det \mathbf{M} \neq 0$. In this case the symmetrical cylindrical framework is both statically and kinematically determinate. If the first l rings are unloaded $(\mathbf{p}_1 = \mathbf{p}_2 = \cdots \mathbf{p}_l = 0;\ l \leqq m)$, then it follows from the structure of the matrix **M** that there are zero forces in the bars of the first l rings, i.e. the forces in the reticulated cylinder spread only downward. It should be noted that a detailed analysis for the symmetrical cylindrical framework is given in[5] for the case of odd n.

If n is even (and we shall deal only with this case for the remainder of this paper) then $\det \mathbf{L} = 0$ and so on the basis of eqn (24) we have $\det \mathbf{M} = 0$. Since the matrix **M** is square, the singularity of **M** means that the symmetrical cylindrical framework is both statically and kinematically indeterminate. The question is how the degree of indeterminacy will increase in comparison with single degree of indeterminacy of a ring, since the cylinder was obtained by building one unstable ring on another. Another question is what is the physical meaning of the static and kinematic indeterminacy in the case of a cylinder. These problems will now be analyzed.

Let us consider those equations of the set (19) which are determined by the jth row of submatrices of **M**. Let us add the $3k$-2nd equations $(k = 2,3,\ldots,n)$ in the jth row of submatrices to the first equation of the jth row of submatrices if k is odd, and let us subtract them if k is even. In this manner the elements of the first row of matrix **L** in the jth row of submatrices will be equal to zero and the first row of the matrix **N** will be as follows:

$$\overbrace{0 \quad b \quad -b}^{1} \quad \overbrace{0 \quad -b \quad b}^{2} \quad \overbrace{0 \quad b \quad -b}^{3} \quad \ldots \quad \overbrace{0 \quad b \quad -b}^{n-1} \quad \overbrace{0 \quad -b \quad b}^{n}. \tag{25}$$

We find that the first element of the load vector $\mathbf{p}_j$ is

$$P'_{(j-1)n+1,(j-1)n+2} = \sum_{k=1}^{n-1} (-1)^{k+1} P_{(j-1)n+k,(j-1)n+k+1} - P_{jn,(j-1)n+1}. \tag{26}$$

The remaining rows of the matrices **L**, **N** and the rest of the elements of the vector $\mathbf{p}_j$ do not change. Let us do this transformation in all the rows of submatrices $(j = 1,2,\ldots,m)$. Let us denote the transformed form of the matrices **L**, **N** and the vector $\mathbf{p}_j$ by $\mathbf{L}'$, $\mathbf{N}'$ and $\mathbf{p}'_j$, respectively:

$$\mathbf{L}' = \begin{array}{|c|l}
\multicolumn{1}{c}{\overbrace{\hphantom{\text{the last } 3n-1}}^{3n}} & \\
\hline
0\ 0\ 0\ \ldots\ 0 & \}\ 1 \\
\hline
\text{the same as} & \\
\text{the last } 3n-1 & \Big\}\ 3n-1 \\
\text{rows of } \mathbf{L} & \\
\hline
\end{array}
\qquad
\mathbf{N}' = \begin{array}{|c|l}
\multicolumn{1}{c}{\overbrace{\hphantom{\text{the last } 3n-1}}^{3n}} & \\
\hline
0\ b\ -b\ \ldots\ 0\ -b\ b & \}\ 1 \\
\hline
\text{the same as} & \\
\text{the last } 3n-1 & \Big\}\ 3n-1 \\
\text{rows of } \mathbf{N} & \\
\hline
\end{array}$$

$$\mathbf{p}'_j = \begin{array}{|c|l}
\multicolumn{1}{c}{\overbrace{\hphantom{\text{the last } 3n-1}}^{1}} & \\
\hline
P'_{(j-1)n+1,(j-1)n+2} & \}\ 1 \\
\hline
\text{the same as} & \\
\text{the last } 3n-1 & \Big\}\ 3n-1 \\
\text{elements of } \mathbf{p}_j & \\
\hline
\end{array}
\qquad j = 1,2,\ldots,m.$$

With this notation we may write eqn (19) in the form:

$$\begin{bmatrix} \mathbf{L}' & & & & & \\ \mathbf{N}' & \mathbf{L}' & & & & \\ & \mathbf{N}' & \mathbf{L}' & & & \\ & & \ddots & \ddots & & \\ & & & \mathbf{N}' & \mathbf{L}' & \\ & & & & \mathbf{N}' & \mathbf{L}' \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \vdots \\ \mathbf{s}_{m-1} \\ \mathbf{s}_m \end{bmatrix} = \begin{bmatrix} \mathbf{p}'_1 \\ \mathbf{p}'_2 \\ \mathbf{p}'_3 \\ \vdots \\ \mathbf{p}'_{m-1} \\ \mathbf{p}'_m \end{bmatrix}. \tag{27}$$

Now, let us change the order of succession of the equations. Let us put the first equation in each row of submatrices of (27) into the place of the first equation of the preceding row of submatrices, and the very first equation into the place of the first equation of the very last row of submatrices. In this manner, we obtain

$$\begin{bmatrix} \mathbf{L}'' & & & & & \\ \mathbf{N}'' & \mathbf{L}'' & & & & \\ & \mathbf{N}'' & \mathbf{L}'' & & & \\ & & \ddots & \ddots & & \\ & & & \mathbf{N}'' & \mathbf{L}'' & \\ & & & & \mathbf{N}'' & \boxed{\mathbf{L}'} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \vdots \\ \mathbf{s}_{m-1} \\ \mathbf{s}_m \end{bmatrix} = \begin{bmatrix} \mathbf{p}''_1 \\ \mathbf{p}''_2 \\ \mathbf{p}''_3 \\ \vdots \\ \mathbf{p}''_{m-1} \\ \mathbf{p}''_m \end{bmatrix} \tag{28}$$

where

$$\mathbf{L}'' = \overbrace{\begin{array}{|c|} \hline 0\ b\ -b \ldots 0\ -b\ b \\ \hline \text{the same as} \\ \text{the last } 3n-1 \\ \text{rows of } \mathbf{L} \\ \hline \end{array}}^{3n} \begin{array}{l} \}\,1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3n-1 \end{array} \qquad \mathbf{N}'' = \overbrace{\begin{array}{|c|} \hline 0\ 0\ 0 \ldots 0\ 0 \\ \hline \text{the same as} \\ \text{the last } 3n-1 \\ \text{rows of } \mathbf{N} \\ \hline \end{array}}^{3n} \begin{array}{l} \}\,1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3n-1 \end{array}$$

$$\mathbf{p}''_j = \overbrace{\begin{array}{|c|} \hline P'_{jn+1,\,jn+2} \\ \hline \text{the same} \\ \text{as the last} \\ 3n-1 \text{ elements} \\ \text{of } \mathbf{p}_j \\ \hline \end{array}}^{1} \begin{array}{l} \}\,1 \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 3n-1 \end{array} \qquad \mathbf{p}''_m = \overbrace{\begin{array}{|c|} \hline P'_{1,2} \\ \hline \text{the same} \\ \text{as the last} \\ 3n-1 \text{ elements} \\ \text{of } \mathbf{p}_m \\ \hline \end{array}}^{1} \begin{array}{l} \}\,1 \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} 3n-1 \end{array}$$

$j = 1, 2, \ldots, m-1.$

It may be seen that the first rows of $\mathbf{L}'$ and $\mathbf{N}'$ changed places and the right lower submatrix of the coefficient matrix of eqn (27) remained unchanged in (28), too. In the first row of the last (mth) row of submatrices in the coefficient matrix of eqn (28), the elements are equal to zero. This set of linear equations does not contain a contradiction provided that in this row, the right-hand side is also equal to zero, i.e. if

$$P'_{1,2} = \sum_{k=1}^{n-1} (-1)^{k+1} P_{k,k+1} - P_{n,1} = 0. \tag{29}$$

We can also interpret the eqn (29) physically, if it is multiplied by the radius of the inscribed circle of the n-gon. In this form, condition (29) expresses the fact that the moments of the horizontal components of the external forces acting on the top of the uppermost ring with respect to the axis of the cylinder, with alternate sign, are in equilibrium.

Let us now analyze the magnitude of the nullity of the coefficient matrix of eqn (28). For this purpose, let us clear the first row of the last row of submatrices and the first column of the last column of submatrices of the coefficient matrix of eqn (28). Let us denote the resulting minor matrix by $\mathbf{M}_1$, which is of the following form:

$$\begin{array}{cc} & \begin{array}{cccccc} \underbrace{1} & \underbrace{2} & \underbrace{3} & \cdots & \underbrace{m-1} & \underbrace{m} \end{array} \\ \mathbf{M}_1 = & \begin{bmatrix} \mathbf{L}'' & & & & & \\ \mathbf{N}'' & \mathbf{L}'' & & & & \\ & \mathbf{N}'' & \mathbf{L}'' & & & \\ & & \ddots & \ddots & & \\ & & & \mathbf{N}'' & \mathbf{L}'' & \\ & & & & \mathbf{N}''' & \mathbf{L}_{11} \end{bmatrix} \end{array} \tag{30}$$

where

$$\mathbf{N}''' = \left.\overbrace{\boxed{\begin{array}{l} \text{the same as} \\ \text{the last } 3n-1 \\ \text{rows of } \mathbf{N} \end{array}}}^{3n}\right\} 3n-1$$

and $\mathbf{L}_{11}$ is the main minor matrix generated by the first element of the first row of the matrix $\mathbf{L}$. It is easy to demonstrate that

$$\det \mathbf{L}_{11} = 1 \tag{31}$$

and

$$\det \mathbf{L}'' = -2(n-1)ab. \tag{32}$$

Since

$$\det \mathbf{M}_1 = (\det \mathbf{L}'')^{m-1} \cdot (\det \mathbf{L}_{11}), \tag{33}$$

and since neither $\det \mathbf{L}_{11}$ nor $\det \mathbf{L}''$ is equal to zero, on account of expressions (31), (32), it follows that $\det \mathbf{M}_1 \neq 0$. The rank of the matrix $\mathbf{M}_1$ is $3nm-1$. Therefore the rank of the coefficient matrix of eqn (28), and at the same time, of eqn (19) also is equal to $3nm-1$, i.e. the nullity of the matrix is 1. Thus we discover that the symmetrical cylindrical framework is both statically and kinematically *singly-indeterminate, independently of the number of the rings in the cylinder.* Accordingly, the degree of static and kinematic indeterminacy does not increase if the number of the rings is increased. The rings, which are individually unstable, thus make each other stiff by building one on top of the other.

In the case of single ring the static indeterminacy and the kinematic indeterminacy appeared together. But in the case of a cylinder these two properties separate; that is, they do not appear in a single ring (Fig. 7). The single degree of static indeterminacy for a cylinder appears in such a way that the lowest ring, which is attached to the foundation, is statically indeterminate. This results from the fact that we can clear the first column of the last column of submatrices of the coefficient matrix of eqn (28) by taking this column of the set of equations to the right-hand side with opposite sign. In this manner the solution gives a one -parameter system of forces in the bars. The static indeterminacy, however, is not associated with the part of the cylinder, which is above the *lowest* ring, since the part of eqn (28) which can be obtained by clearing the last row

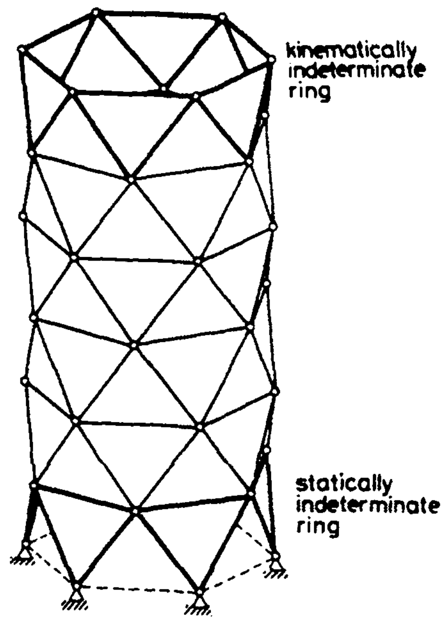


Fig. 7. The kinematically indeterminate ring and the statically indeterminate ring in the cylinder.

of submatrices (and the last column of submatrices) is solvable independently of the cleared part; and uniqueness is valid for the solution. It can be stated that a system of forces which is in a state of self-stress can be added to the forces in the bars of the lowest ring without disturbing equilibrium.

The single-degree of kinematic indeterminacy of the cylinder appears in the *uppermost* ring, since the condition of freedom from contradiction with respect to eqn (19) applies to the forces acting on the top of the uppermost ring. If the forces acting on the top of the uppermost ring do not satisfy the condition (29), then equilibrium is not possible and the uppermost ring begins to move in the manner detailed in Section 2. The uppermost ring thus constitutes a mechanism. As the degree of kinematic indeterminacy of the entire cylinder is 1 and this single-degree of kinematic indeterminacy appears in the mechanism of the uppermost ring, it follows that the remaining part of the structure is kinematically determinate, i.e. rigid.

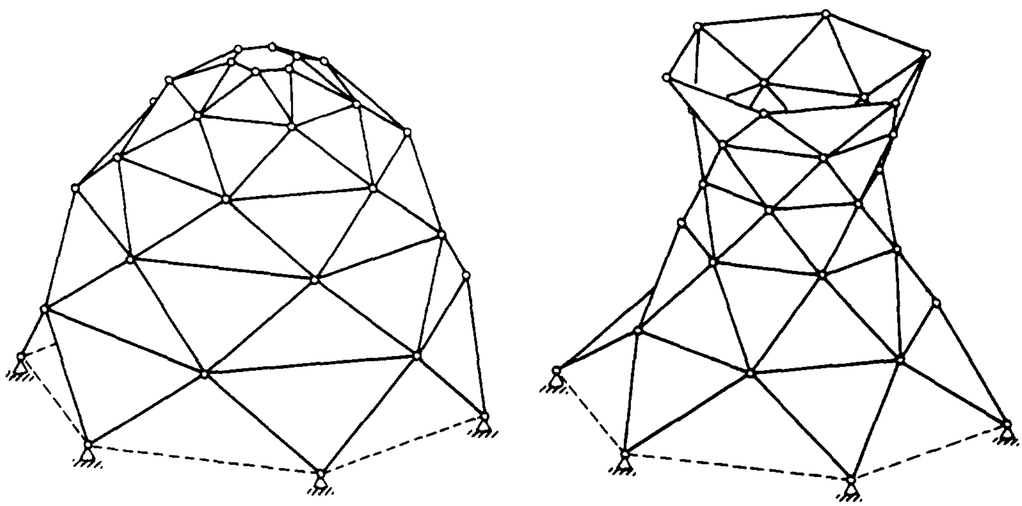
When n is even and the first l rings are unloaded then the magnitude of the forces in bars in the first $l-1$ rings will be equal to zero and in the lth ring, the forces in bars may form a state of self-stress. The forces in the cylinder spread downward and also one ring upward. This statement can be proved by eqn (28).

4. SUPPLEMENTARY REMARKS

The importance of the results obtained for the symmetrical cylindrical lattice is that they can be generalized for towers having a symmetrical lattice fitted to an arbitrary surface of revolution (Fig. 8).

The static and kinematic indeterminacy is independent of the height of the rings. This can be seen from eqn (11). If the side-lengths of the upper and lower n-gons of a ring are different, then the joints of the ring may lie on the surface of a regular cone. It can be proved (but we do not give a proof here) that in this case the properties of the ring with respect to the static and kinematic indeterminacy will not change in comparison with a ring fitted to a cylinder of revolution. If a symmetrically reticulated dome or tower is composed of rings fitted to a regular conical surface with different generatrix slopes (see Fig. 8), then the structure of the coefficient matrix of the set of equilibrium equations will be the same as that of the matrix **M**, but in this case the submatrices appearing in both the main and accessory diagonals will be different from each other.

In the case of even n, it can be verified that the degree of static and kinematic indeterminacy of a symmetrically reticulated structure fitted to an arbitrary surface of revolution is generally



(a) (b)

Fig. 8. Single-layer symmetrical space trusses fitted to (a) elliptic and (b) hyperbolic surfaces of revolution.

equal to 1. For trusses of the type shown in Fig. 8(b), however, special cases can occur. If in common joints of two connecting rings the joining inclined bars lie in a common plane, then the degree of static and kinematic indeterminacy increases. A simple example is the case in which the form of the structure is such that the inclined bars lie on the generatrices of a hyperboloid of revolution.

In "Tensegrity" structures, and similarly in cable nets, prestress gives stiffness for the structure[3]. The kinematic indeterminacy at these structures appears in a form of "infinitesimal mechanisms". For the symmetrical cylindrical framework. on the other hand, the kinematic indeterminacy appears in a form of a "large displacement mechanism"; and in this case it seems that prestress does not give stiffness for the truss.

5. CONCLUSIONS

For a cylindrical truss consisting of congruent rings with cyclic symmetry, having no planes of mirror symmetry, there exists such a geometry of the truss, for which the truss is both statically and kinematically indeterminate, independently of the parity of the side-number n of the polygon in the ring. This is a "Tensegrity" cylinder. In this case, the individual rings in the cylinder are both statically and kinematically indeterminate and so the degree of indeterminacy is equal to the number of the rings. The physical meaning of this fact is that, for these cylinders, there exist as many linearly independent states of self-stress and "infinitesimal mechanisms", respectively, as rings are in the cylinder.

For a cylindrical truss consisting of congruent rings with cyclic symmetry, having planes of mirror symmetry, the truss is both statically and kinematically determinate, if n is odd. If n is even, then the truss is both statically and kinematically indeterminate and the degree of indeterminacy is always 1, independently of the number of the rings in the cylinder. The single degree of static indeterminacy means that the lowest ring of the cylinder is in a one-parameter state of self-stress. The single degree of kinematic indeterminacy means that the uppermost ring is a "large displacement mechanism" of single degree of freedom.

Conclusions for symmetrical cylindrical trusses are also valid for symmetrically reticulated structures fitted to elliptic and parabolic surfaces of revolution. But in the hyperbolic case when n is even, the degree of both static and kinematic indeterminacy can be greater than 1. In the structure, "infinitesimal mechanisms" can arise in addition to the "large displacement mechanism" of the uppermost ring.

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